

SPECTRAL SHIFT FUNCTION FOR OPERATORS WITH CROSSED MAGNETIC AND ELECTRIC FIELDS

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ABSTRACT. We obtain a representation formula for the derivative of the spectral shift function $\xi(\lambda; B, \epsilon)$ related to the operators $H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x$ and $H(B, \epsilon) = H_0(B, \epsilon) + V(x, y)$, $B > 0, \epsilon > 0$. We establish a limiting absorption principle for $H(B, \epsilon)$ and an estimate $\mathcal{O}(\epsilon^{n-2})$ for $\xi'(\lambda; B, \epsilon)$, provided $\lambda \notin \sigma(Q)$, where $Q = (D_x - By)^2 + D_y^2 + V(x, y)$.

1. INTRODUCTION

Consider the two-dimensional Schrödinger operator with homogeneous magnetic and electric fields

$$H = H(B, \epsilon) = H_0(B, \epsilon) + V(x, y), \quad D_x = -i\partial_x, \quad D_y = -i\partial_y,$$

where

$$H_0 = H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x.$$

Here $B > 0$ and $\epsilon > 0$ are proportional to the strength of the homogeneous magnetic and electric fields. We assume that $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ and $V(x, y)$ satisfies the estimate

$$|V(x, y)| \leq C(1 + |x|)^{-2-\delta}(1 + |y|)^{-1-\delta}, \quad \delta > 0. \quad (1.1)$$

For $\epsilon \neq 0$ we have $\sigma_{\text{ess}}(H_0(B, \epsilon)) = \sigma_{\text{ess}}(H(B, \epsilon)) = \mathbb{R}$. On the other hand, for decreasing potentials V we may have embedded eigenvalues $\lambda \in \mathbb{R}$ and this situation is completely different from that with $\epsilon = 0$ when the spectrum of $H(B, 0)$ is formed by eigenvalues with finite multiplicities which may accumulate only to Landau levels $\lambda_n = (2n + 1)B$, $n \in \mathbb{N}$ (see [9], [13], [15] and the references cited there). The spectral properties of H and the existence of resonances have been studied in [7], [8], [5] under the assumption that $V(x, y)$ admits a holomorphic extension in the x -variable into a domain

$$\Gamma_{\delta_0} = \{z \in \mathbb{C} : 0 \leq |\text{Im } z| \leq \delta_0\}.$$

Moreover, without any assumption on the analyticity of $V(x, y)$ we show in Proposition 2 below that the operator $(H - z)^{-1} - (H_0 - z)^{-1}$ for $z \in \mathbb{C}$, $\text{Im } z \neq 0$, is trace class and following the general setup [11], [20], we define the spectral shift function $\xi(\lambda) = \xi(\lambda; B, \epsilon)$ related to $H_0(B, \epsilon)$ and $H(B, \epsilon)$ by

$$\langle \xi', f \rangle = \text{tr} \left(f(H) - f(H_0) \right), \quad f \in C_0^\infty(\mathbb{R}).$$

By this formula $\xi(\lambda)$ is defined modulo a constant but for the analysis of the derivative $\xi'(\lambda)$ this is not important. Moreover, the above property of the resolvents and Birman-Kuroda theorem imply $\sigma_{\text{ac}}(H_0(B, \epsilon)) = \sigma_{\text{ac}}(H(B, \epsilon)) = \mathbb{R}$. A representation of the derivative $\xi'(\lambda; B, \epsilon)$ has been obtained in [5] for strong magnetic fields $B \rightarrow +\infty$ under the assumption that $V(x, y)$ admits an analytic

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continuation in x -direction. Moreover, the distribution of the resonances z_j of the perturbed operator $H(B, \epsilon)$ has been examined in [5] and a Breit-Wigner representation of $\xi'(\lambda; B, \epsilon)$ involving the resonances z_j was established.

In the literature there are a lot of works concerning Schrödinger operators with magnetic fields ($\epsilon = 0$) but there are only few ones dealing with magnetic and Stark potentials ($\epsilon \neq 0$) (see [7], [8], [5] and the references given there). It should be mentioned that the tools in [7], [8] and [5] are related to the resonances of the perturbed problem and to define the resonances one supposes that the potential $V(x, y)$ has an analytic continuation in x variable. In this paper we consider the operator H without *any assumption* on the analytic continuation of $V(x, y)$ and without the *restriction* $B \rightarrow +\infty$. Our purpose is to study $\xi'(\lambda; B, \epsilon)$ and the existence of embedded eigenvalues of H . To examine the behavior of the spectral shift function we need a representation of the derivative $\xi'(\lambda; B, \epsilon)$. The key point in this direction is the following

Theorem 1. *Let $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ and let (1.1) hold for V and $\partial_x V$. Then for every $f \in C_0^\infty(\mathbb{R})$ and $\epsilon \neq 0$ we have*

$$\mathrm{tr} \left(f(H) - f(H_0) \right) = -\frac{1}{\epsilon} \mathrm{tr} \left(\partial_x V f(H) \right). \quad (1.2)$$

The formula (1.2) has been proved by D. Robert and X.P.Wang [18] for Stark Hamiltonians in absence of magnetic field ($B = 0$). In fact, the result in [18] says that

$$\xi'(\lambda; 0, \epsilon) = -\frac{1}{\epsilon} \int_{\mathbb{R}^2} \partial_x V \frac{\partial e}{\partial \lambda}(x, y, x, y; \lambda, 0, \epsilon) dx dy, \quad (1.3)$$

where $e(\cdot, \cdot; \lambda, 0, \epsilon)$ is the spectral function of $H(0, \epsilon)$. The presence of magnetic field $B \neq 0$ and Stark potential lead to some serious difficulties. The operator H is not elliptic for $|x| + |y| \rightarrow \infty$ and we have double characteristics. On the other hand, the commutator $[H, x]$ involves the term $(D_x - By)$ and it creates additional difficulties. The proof of Theorem 1 is long and technical. We are going to study the trace class properties of the operators $\psi(H \pm \mathbf{i})^{-N}$, $\partial_x \circ \psi(H \pm \mathbf{i})^{-N-1}$, $(H \pm \mathbf{i})\partial_x \circ \psi(H \pm \mathbf{i})^{-N-2}$ etc. for $N \geq 2$ and $\psi \in C_0^\infty(\mathbb{R}^2)$ (see Lemmas 1 and 2). Moreover, by an argument similar to that in Proposition 2.1 in [5], we obtain estimates for the trace norms of the operators

$$(z - H)^{-1} V (z' - H)^{-1}, V (z - H)^{-1} (z' - H)^{-1}, z \notin \mathbb{R}, z' \notin \mathbb{R}$$

and we apply an approximation argument. Notice that in [18] the spectral shift function is related to the trace of the *time delay* operator $T(\lambda)$ defined via the corresponding scattering matrix $S(\lambda)$ (see [17]). In contrast to [18], our proof is direct and neither $T(\lambda)$ nor $S(\lambda)$ corresponding to the operator $H(B, \epsilon)$ are used.

The second question examined in this work is the existence of embedded real eigenvalues and the limiting absorption principle for H . In the physical literature one conjectures that for $\epsilon \neq 0$ there are no embedded eigenvalues. We establish in Section 3 a weaker result saying that in any interval $[a, b]$ we may have at most a finite number embedded eigenvalues with finite multiplicities. Under the assumption for analytic continuation of V it was proved in [7] that for some finite interval $[\alpha(B, \epsilon), \beta(B, \epsilon)]$ there are no resonances z of $H(B, \epsilon)$ with $\mathrm{Re} z \notin [\alpha(B, \epsilon), \beta(B, \epsilon)]$. Since the real resonances z coincide with the eigenvalues of $H(B, \epsilon)$, we obtain some information for the embedded eigenvalues. On the other hand, exploiting the analytic continuation and the resonances we proved in [5] that for $B \rightarrow +\infty$ the real parts $\mathrm{Re} z_j$ of the resonances z_j lie outside some

neighborhoods of the Landau levels. Thus the Landau levels play a role in the distribution of the resonances. It is known that the spectrum of the operator $Q = (D_x - By)^2 + D_y^2 + V(x, y)$ with decreasing potential V is formed by eigenvalues (see [9], [13], [15]). In this paper we establish a limiting absorption principle for $\lambda \notin \sigma(Q)$. In particular, we show that there are no embedded eigenvalues outside $\sigma(Q)$. This agrees with the result in [5] obtained under the restrictions on the behavior of V and $B \rightarrow +\infty$. On the other hand, the result of Proposition 3 and the estimates (4.3) have been established by X. P. Wang [19] for Stark operators with $B = 0$.

Following the results in Section 4 and the representation of $\xi'(\lambda; B, \epsilon)$ given in [5], it is natural to expect that for $\lambda \notin \sigma(Q)$ the derivative of the spectral shift function $\xi'(\lambda; B, \epsilon)$ must be bounded. In fact, we prove the following stronger result.

Theorem 2. *Let the potential $V \in C^\infty(\mathbb{R}^2; \mathbb{R})$ satisfy with some $\delta > 0$ and $n \in \mathbb{N}$, $n \geq 2$ the estimates*

$$|\partial_x^\alpha \partial_y^\beta V(x, y)| \leq C_{\alpha, \beta} (1 + |x|)^{-n-\delta-|\alpha|} (1 + |y|)^{-2-\delta-|\beta|}, \quad \forall \alpha, \forall \beta. \quad (1.4)$$

Then for $\lambda_0 \notin \sigma(Q)$ we have

$$\xi'(\lambda; B, \epsilon) = \mathcal{O}(\epsilon^{n-2}) \quad (1.5)$$

uniformly for λ in a small neighborhood $\Xi \subset \mathbb{R}$ of λ_0 .

The estimate (1.5) has been obtained in [18] in the case of absence of magnetic field $B = 0$ (for a Breit-Wigner formula see [10], [4] for Stark Hamiltonians and [5] for the operator $H(B, \epsilon)$). Our approach is quite different from that in [18]. Our proof is going without an application of a representation similar to (1.3) which leads to complications connected with the behavior of the spectral function $e(\cdot, \cdot; \lambda, B, \epsilon)$ corresponding to $H(B, \epsilon)$. The formula (1.2) plays a crucial role and our analysis is based on a complex analysis argument combined with a representation of $f(H)$ involving the almost analytic continuation of $f \in C_0^\infty(\mathbb{R})$. In this direction our argument is similar to that developed in [4] and [5].

The plan of this paper is as follows. In Sect. 2 we establish Theorem 1. The embedded eigenvalues and Mourre estimates are examined in Sect. 3. In Sect. 4 we prove Proposition 3 concerning the limiting absorption principle for $H(B, \epsilon)$. Finally, in Sect. 5 we establish Theorem 2.

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2. REPRESENTATION OF THE SPECTRAL SHIFT FUNCTION

Throughout this work we will use the notations of [3] for symbols and pseudodifferential operators. In particular, if $m : \mathbb{R}^4 \rightarrow [0, +\infty[$ is an order function (see [3], Definition 7.4), we say that $a(z, \zeta) \in S^0(m)$ if for every $\alpha \in \mathbb{N}^4$ there exists $C_\alpha > 0$ such that

$$|\partial_{z, \zeta}^\alpha a(z, \zeta)| \leq C_\alpha m(z, \zeta).$$

In the special case when $m = 1$, we will write S^0 instead of $S^0(1)$. We will use the standard Weyl quantization of symbols. More precisely, if $p(z, \zeta)$, $(z, \zeta) \in \mathbb{R}^4$, is a symbol in $S^0(m)$, then

$P^w(z, D_z)$ is the operator defined by

$$P^w(z, D_z)u(z) = (2\pi)^{-2} \iint e^{i(z-z')\cdot\zeta} p\left(\frac{z+z'}{2}, \zeta\right) u(z') dz' d\zeta, \quad \text{for } u \in \mathcal{S}(\mathbb{R}^2).$$

We denote by $P^w(z, hD_z)$ the semiclassical quantization obtained as above by quantizing $p(z, h\zeta)$.

Our goal in this section is to prove Theorem 1. For this purpose we need some Lemmas. We set

$$Q_0 = H_0 - \epsilon x = (D_x - By)^2 + D_y^2, \quad Q = Q_0 + V,$$

and in Lemma 1 we will use the notation $H_1 = H$. For the simplicity we assume that $\epsilon = B = 1$. The general case can be covered by the same argument.

Lemma 1. *Assume that $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ and let $\psi \in C_0^\infty(\mathbb{R}^2)$. Then for $N \geq 2$, $j = 0, 1$ and for $\text{Im } z \neq 0$, the following operators are trace class:*

$$i) \psi(H_j \pm \mathbf{i})^{-N}, \quad \partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-1}, \quad (H_j \pm \mathbf{i}) \partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-2}.$$

$$ii) (H_j \pm \mathbf{i})^{-N} \psi, \quad (H_j \pm \mathbf{i})^{-N-1} \psi \cdot \partial_x.$$

$$iii) \psi \circ \partial_x (H_j \pm \mathbf{i})^{-N-1}, \quad (H_j \pm \mathbf{i}) \psi \circ \partial_x (H_j \pm \mathbf{i})^{-N-2}.$$

$$iv) (H_j \pm \mathbf{i}) \partial_x (H_j \pm \mathbf{i})^{-N-2} \psi.$$

$$v) (H_1 + \mathbf{i}) \partial_x (H_1 + \mathbf{i})^{-N-1} (H_1 - z)^{-1} \psi.$$

Moreover,

$$\|(H_1 + \mathbf{i}) \partial_x (H_1 + \mathbf{i})^{-N-1} (H_1 - z)^{-1} \psi\|_{\text{tr}} = \mathcal{O}\left(\frac{|z| + 1}{|\text{Im } z|^2}\right). \quad (2.1)$$

Proof. We will prove the lemma only for $(H_1 + \mathbf{i})$, the case concerning $(H_1 - \mathbf{i})$ is similar. On the other hand, the statements for $(H_0 + \mathbf{i})$ follow from those for $(H_1 + \mathbf{i})$ when $V = 0$.

From the first resolvent equation, we obtain

$$\begin{aligned} (H_1 + z)^{-1} &= (Q_0 + z)^{-1} - (Q_0 + z)^{-1} (x + V) (H_1 + z)^{-1} \\ &= (Q_0 + z)^{-1} + \sum_{j=1}^{N+2} (-1)^j (Q_0 + z)^{-1} \left((x + V) (Q_0 + z)^{-1} \right)^j \\ &\quad + (-1)^{N+3} \left((Q_0 + z)^{-1} (x + V) \right)^{N+3} (H_1 + z)^{-1}. \end{aligned} \quad (2.2)$$

Taking $(N-1)$ derivatives with respect to z in the above identity and setting $z = \mathbf{i}$, we see that $(H_1 + \mathbf{i})^{-N}$ is a linear combination of terms

$$\mathcal{K}_N := (Q_0 + \mathbf{i})^{-j_1} W (Q_0 + \mathbf{i})^{-j_2} W \dots (Q_0 + \mathbf{i})^{-j_r} W (H_1 + \mathbf{i})^{-p},$$

with $j_1 + \dots + j_r \geq N$, $j_1 \geq 1$, $p \geq 0$ and $W(x) = x + V(x)$.

Recall that if $P \in S^0(m)$ with $m \in L^1(\mathbb{R}^4)$, (resp. $m \in L^2(\mathbb{R}^4)$) then the corresponding operator is trace class (resp. Hilbert-Schmidt). By using this and the fact that the symbol of $(Q_0 + \mathbf{i})^{-1}$ is in $S^0(\langle \xi - y, \eta \rangle^{-2})$, we deduce that the operator

$$K_{l,p,l',p'}^j := \langle x \rangle^{-l} \langle y \rangle^{-p} (Q_0 + \mathbf{i})^{-j} \langle x \rangle^{l'} \langle y \rangle^{p'}$$

is trace class one for $l - l', p - p' > 1, j \geq 2$ and Hilbert-Schmidt one for $l - l', p - p' > 1/2, j \geq 1$.

Next, we write $\psi \mathcal{K}_N$ as follows

$$\begin{aligned} \psi \mathcal{K}_N &= \psi \langle x \rangle^{3r} \langle y \rangle^{2r} K_{3r, 2r, 3r-2, 2r-2}^{j_1} W \langle x \rangle^{-1} K_{3r-3, 2r-2, 3r-1, 2r-4}^{j_2} W \langle x \rangle^{-1} \\ &\quad \dots W \langle x \rangle^{-1} \mathcal{K}_{3, 2, 1, 0}^{j_r} W \langle x \rangle^{-1} (H_1 + \mathbf{i})^{-p} \end{aligned} \quad (2.3)$$

Since $j_1 + j_2 + \dots + j_r \geq N \geq 2$, in the above decomposition, there are at least two Hilbert-Schmidt operators or one of trace class. Combining this with the fact $\psi\langle x \rangle^{3r}\langle y \rangle^{2r}$, $W\langle x \rangle^{-1}$ and $(H_1 + \mathbf{i})^{-p}$ are bounded from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, we conclude that $\psi\mathcal{K}_N$ is trace class operator. Thus $\psi(H_1 + \mathbf{i})^{-N}$ is also a trace class operator. Repeating the same arguments, we obtain the proof for $\partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-1}$.

As above to treat $(H_j \pm \mathbf{i})\partial_x \circ \psi(H_j \pm \mathbf{i})^{-N-2}$, it suffices to show that $(H_j \pm \mathbf{i})\partial_x \circ \psi\mathcal{K}_N$ is trace class. If we have $j_1 \geq 2$ the proof is completely similar to that of $\psi(H_1 + \mathbf{i})^{-N}$. In the case where $j_1 = 1$ since $(H_1 + \mathbf{i})\partial_x(Q_0 + \mathbf{i})^{-1}$ is not bounded, we have to exploit the following representation

$$\begin{aligned} (H_1 + \mathbf{i})\partial_x \circ \psi\mathcal{K}_N &= (H_1 + \mathbf{i})(\partial_x \psi)\mathcal{K}_N \\ &+ (H_1 + \mathbf{i})\psi(Q_0 + \mathbf{i})^{-1}\partial_x \circ W(Q_0 + \mathbf{i})^{-j_2}W\dots(Q_0 + \mathbf{i})^{-j_r}W(H_1 + \mathbf{i})^{-p}. \end{aligned}$$

Next use the fact that $\partial_x W \in L^\infty$ and repeat the argument of the proof above.

Recall that A is trace class if and only if the adjoint operator A^* is trace class. Consequently, (i) implies (ii). Since $\psi \cdot \partial_x = \partial_x \cdot \psi - (\partial_x \psi)$, the assertion (iii) follows from (i).

To deal with (iv), we apply the following obvious identity with $z = -\mathbf{i}$,

$$\partial_x(H - z)^{-1} = (H - z)^{-1}\partial_x + (H - z)^{-1}(1 + \partial_x V)(H - z)^{-1}, \quad (2.4)$$

and obtain

$$(H_1 + \mathbf{i})\partial_x(H_1 + \mathbf{i})^{-N}\psi = (H_1 + \mathbf{i})^{-N}\partial_x \psi + \sum_{j=0}^{N-1} (H_1 + \mathbf{i})^{-j}(1 + \partial_x V)(H_1 + \mathbf{i})^{-N+j}\psi. \quad (2.5)$$

Applying (i) and (ii) to each term on the right hand side of (2.5), we get (iv).

Now we pass to the proof of (v). Applying (2.4), we obtain

$$\begin{aligned} (H_1 + \mathbf{i})\partial_x(H_1 + \mathbf{i})^{-N-1}(H_1 - z)^{-1}\psi &= (H_1 + \mathbf{i})(H_1 - z)^{-1}\partial_x(H_1 + \mathbf{i})^{-N-1}\psi \\ &+ (H_1 + \mathbf{i})(H_1 - z)^{-1}(1 + \partial_x V)(H_1 - z)^{-1}(H_1 + \mathbf{i})^{-N}\psi. \end{aligned}$$

Combining the above equation with (i), (ii), (iv) and using the estimate

$$\|(H_1 + \mathbf{i})(H_1 - z)^{-1}\| = \mathcal{O}\left(\frac{|z| + 1}{|\operatorname{Im} z|}\right),$$

we get (2.1). □

Lemma 2. Assume that $V(x, y) = \phi(x, y)W(x, y)$, where $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ and $W, \partial_x W \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$. Then for $N \geq 4$ the operator

$$(H + \mathbf{i})\partial_x \left[(H + \mathbf{i})^{-N} - (H_0 + \mathbf{i})^{-N} \right],$$

is trace class.

Proof. Taking $(N - 1)$ derivatives with respect to z in the resolvent identity

$$(H + z)^{-1} - (H_0 + z)^{-1} = -(H + z)^{-1}V(H_0 + z)^{-1}$$

and setting $z = \mathbf{i}$, we see that $(H + \mathbf{i})^{-N} - (H_0 + \mathbf{i})^{-N}$ is a linear combination of terms

$$(H + \mathbf{i})^{-j}V(H_0 + \mathbf{i})^{-(N+1+j)}$$

with $1 \leq j \leq N$. Composing the above terms by $(H + \mathbf{i})\partial_x$ and applying Lemma 1, we complete the proof. □

Lemma 3. *Assume that V satisfies the assumptions of Lemma 1. Let $f \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^2)$. Then the operators*

$$\psi f(H_i), \quad H_i \psi \partial_x f(H_i), \quad \psi \partial_x H_i f(H_i)$$

are trace class and we have

$$\mathrm{tr} \left(H_i \psi \partial_x f(H_i) \right) = \mathrm{tr} \left(\psi \partial_x H_i f(H_i) \right).$$

Proof. Set $g(x) = (x + \mathbf{i})^4 f(x)$. Since $g(H_i)$ is bounded, it follows from Lemma 1 that the operators

$$\psi(H_i + \mathbf{i})^{-4} g(H_i), \quad H_i \psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i), \quad \psi \partial_x (H_i + \mathbf{i})^{-4} H_i g(H_i),$$

are trace class, and the cyclicity of the trace yields

$$\begin{aligned} \mathrm{tr} \left(H_i \psi \partial_x f(H_i) \right) &= \mathrm{tr} \left(H_i \psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i) \right) = \mathrm{tr} \left(H_i g(H_i) \psi \partial_x (H_i + \mathbf{i})^{-4} \right) \\ &= \mathrm{tr} \left(\psi \partial_x (H_i + \mathbf{i})^{-4} g(H_i) H_i \right) = \mathrm{tr} \left(\psi \partial_x H_i f(H_i) \right). \end{aligned}$$

Notice that in the above equalities we have used the fact that the operators $g(H_i)$, H_i and $(H_i + \mathbf{i})^{-4}$ commute. \square

Lemma 4. *Let V be as in Lemma 2. Then for every $f \in C_0^\infty(\mathbb{R})$ the operators*

$$f(H) - f(H_0), \quad \partial_x \left(f(H) - f(H_0) \right) \quad \text{and} \quad (H \pm \mathbf{i}) \partial_x \left(f(H) - f(H_0) \right)$$

are trace class.

Proof. Let $g(x) = (x + \mathbf{i})^4 f(x)$ be as above. We decompose

$$\begin{aligned} (H + \mathbf{i}) \partial_x \left(f(H) - f(H_0) \right) &= (H + \mathbf{i}) \partial_x \left((H + \mathbf{i})^{-4} - (H_0 + \mathbf{i})^{-4} \right) g(H_0) + \\ &\quad (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} \left(g(H) - g(H_0) \right) = I + II. \end{aligned}$$

According to Lemma 2, the operator I is trace class. To treat II , we use the Helffer-Sjöstrand formula

$$\begin{aligned} (II) &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} \left((z - H)^{-1} - (z - H_0)^{-1} \right) L(dz) \\ &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) (H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} (z - H)^{-1} V (z - H_0)^{-1} L(dz), \end{aligned}$$

where $\tilde{g}(z) \in C_0^\infty(\mathbb{C})$ is an almost analytic continuation of g such that $\bar{\partial} \tilde{g}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$, while $L(dz)$ is the Lebesgue measure on \mathbb{C} . Now applying Lemma 1, (v), we see that the operator

$$(H + \mathbf{i}) \partial_x (H + \mathbf{i})^{-4} (z - H)^{-1} V$$

is trace class. Since $|z|$ is bounded on $\mathrm{supp} \tilde{g}$, we can apply (2.1) to the right hand part of the above equation and combining this with $\bar{\partial} \tilde{g}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$, we deduce that II is trace class. Summing up, we conclude that $(H + \mathbf{i}) \partial_x \left(f(H) - f(H_0) \right)$ is trace class. The same argument works for $(H - \mathbf{i}) \partial_x \left(f(H) - f(H_0) \right)$. The proof concerning $f(H) - f(H_0)$ and $\partial_x \left(f(H) - f(H_0) \right)$ are similar and simpler. \square

To establish Theorem 1, we also need the following abstract result. For the reader convenience we present a proof.

Proposition 1. *Let A be an operator of trace class on some Hilbert space H and let $\{K_n\}$ be sequences of bounded linear operator which converges strongly to $K \in \mathcal{L}(H)$. Then*

$$\lim_{n \rightarrow \infty} \|K_n A - K A\|_{\text{tr}} = 0.$$

Proof. First assume that A is a finite rank operator having the form $A = \sum_{k=1}^m \langle \cdot, \psi_k \rangle \phi_k$, where $\psi_k, \phi_k \in H$. Since

$$\|A\|_{\text{tr}} \leq \sum_{k=1}^m \|\phi_k\| \|\psi_k\|,$$

we have

$$\|(K_n - K)A\|_{\text{tr}} \leq \sum_{k=1}^m \|(K_n - K)\phi_k\| \|\psi_k\| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.6)$$

The general case can be covered by an approximation. Since K_n converges strongly, it follows from the Banach-Streinhaus theorem that $\mu = \sup_n \|K_n\| < \infty$. Let η be an arbitrary positive constant and let A_η be a finite rank operator such that $\|A - A_\eta\|_{\text{tr}} \leq \frac{\eta}{2\mu}$. We have

$$\|(K_n - K)A\|_{\text{tr}} \leq \|(K_n - K)(A - A_\eta)\|_{\text{tr}} + \|(K_n - K)A_\eta\|_{\text{tr}} \leq \eta + \|(K_n - K)A_\eta\|_{\text{tr}}.$$

Next we apply (2.6) for the finite rank operator A_η and obtain

$$\lim_{n \rightarrow \infty} \|(K_n - K)A\|_{\text{tr}} \leq \eta,$$

which implies Proposition 1, since η is arbitrary. \square

Proof of Theorem 1. Assume first that $V = \phi W$ where $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ and $W, \partial_x W \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$. Choose a function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ for $|(x, y)| \leq 1$. For $R > 0$ set

$$\chi_R(x, y) = \chi\left(\frac{x}{R}, \frac{y}{R}\right),$$

and introduce

$$B_R := [\chi_R \partial_x, H]f(H) - [\chi_R \partial_x, H_0]f(H_0).$$

Here $[A, B] = AB - BA$ denotes the commutator of A and B . According to Lemma 3, we have

$$\text{tr} \left([\chi_R \partial_x, H]f(H) \right) = \text{tr} \left([\chi_R \partial_x, H_0]f(H_0) \right) = 0.$$

Thus

$$\text{tr} (B_R) = 0. \quad (2.7)$$

On the other hand, a simple calculus shows that

$$B_R = \chi_R \left([\partial_x, H]f(H) - [\partial_x, H_0]f(H_0) \right) + [\chi_R, H_0] \partial_x \left(f(H) - f(H_0) \right) := B_R^1 + B_R^2, \quad (2.8)$$

where we have used that $[\chi_R, H] = [\chi_R, H_0]$.

Since $[\partial_x, H] = 1 + \partial_x V$ and $[\partial_x, H_0] = 1$, it follows from Lemma 3, Lemma 4 and Proposition 1 that

$$\lim_{R \rightarrow \infty} \text{tr} (B_R^1) = \text{tr} \left(f(H) - f(H_0) \right) + \text{tr} \left(\partial_x V f(H) \right). \quad (2.9)$$

Next we claim that

$$\lim_{R \rightarrow \infty} B_R^2 = 0. \quad (2.10)$$

Using that $[\chi_R, H_0] = \frac{2}{R}(D_x \chi_R)(D_x - y) - \frac{2}{R}(D_y \chi_R)D_y + \frac{1}{R^2}(\Delta \chi_R)$, we decompose B_R^2 as a sum of three terms $B_R^2 = I_R^1 + I_R^2 + I_R^3$, where

$$\begin{aligned} I_R^1 &= -\frac{2}{R}(D_x \chi_R)(D_x - y)\partial_x(f(H) - f(H_0)), \\ I_R^2 &= -\frac{2}{R}(D_y \chi_R)D_y\partial_x(f(H) - f(H_0)), \\ I_R^3 &= \frac{1}{R^2}(\Delta \chi_R)\partial_x(f(H) - f(H_0)). \end{aligned}$$

To treat I_R^1 , we set $Q = H - x$ and write

$$\begin{aligned} I_R^1 &= -\frac{2}{R}(D_x \chi_R)(D_x - y)(Q_0 - \mathbf{i})^{-1}(H - \mathbf{i})\partial_x(f(H) - f(H_0)) \\ &\quad + \frac{2}{R}(D_x \chi_R)[(D_x - y)(Q - \mathbf{i})^{-1}, x]\partial_x(f(H) - f(H_0)) \\ &\quad + \frac{2}{R}x(D_x \chi_R)(D_x - y)(Q - \mathbf{i})^{-1}\partial_x(f(H) - f(H_0)). \end{aligned}$$

The operators $[(D_x - y)(Q - \mathbf{i})^{-1}, x]$ and $(D_x - y)(Q - \mathbf{i})^{-1}$ are bounded, while $\partial_x(f(H) - f(H_0))$ and $(H - \mathbf{i})\partial_x(f(H) - f(H_0))$ are trace class operators (see Lemma 4). On the other hand, $\frac{2}{R}(D_x \chi_R)$, $\frac{2}{R}x(D_x \chi_R)$ converges strongly to zero. Indeed, since $\chi(x, y) = 1$ for $|(x, y)| \leq 1$, we get

$$\int \left| \frac{x}{R}(D_x \chi_R)u \right|^2 dx dy \leq \sup_{(x, y) \in \mathbb{R}^2} |x D_x \chi(x, y)| \int_{\{|(x, y)| \geq R\}} |u|^2 dx dy \rightarrow 0, \quad R \rightarrow \infty,$$

for all $u \in L^2(\mathbb{R}^2)$. Applying Proposition 1, we conclude that

$$\lim_{R \rightarrow \infty} I_R^1 = 0. \quad (2.11)$$

To deal with I_R^2, I_R^3 , notice that the operators $D_y(Q - \mathbf{i})^{-1}$ and $[D_y(Q - \mathbf{i})^{-1}, x]$ are bounded and we repeat the above argument. Thus we deduce

$$\lim_{R \rightarrow \infty} I_R^j = 0, \quad j = 2, 3. \quad (2.12)$$

Consequently, (2.11) and (2.12) imply (2.10) and the claim is proved. Now, combining (2.7), (2.8), (2.9) and (2.10), we obtain Theorem 1 in the case where V satisfies the assumption of Lemma 2 and $\epsilon = 1$.

Proposition 2. *Assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfies (1.1). Then for $z \notin \mathbb{R}$, $z' \notin \mathbb{R}$ the operators $(z - H)^{-1}V(z' - H)^{-1}$, $V(z - H)^{-1}(z' - H)^{-1}$, $(H - z)^{-1} - (H_0 - z)^{-1}$ are trace class and*

$$\begin{aligned} \|(z - H)^{-1}V(z' - H)^{-1}\|_{\text{tr}} &\leq C_1 |\text{Im } z|^{-1} |\text{Im } z'|^{-1}, \\ \|V(z - H)^{-1}(z' - H)^{-1}\|_{\text{tr}} &\leq C_1 |\text{Im } z|^{-1} |\text{Im } z'|^{-1}. \end{aligned} \quad (2.13)$$

Moreover, if $g \in C_0^\infty(\mathbb{R})$, then the operator $Vg(H)$ is trace class.

Proof. Set $g_\delta(x, y) = \langle x \rangle^{-1-\frac{\delta}{2}} \langle y \rangle^{-\frac{1+\delta}{2}}$ and $f_\delta(x, y) = \langle x \rangle^{-2-\delta} \langle y \rangle^{-1-\delta}$, where δ is the constant in (1.1). According to Lemma 8 in the Appendix, $g_\delta(H_0 + \mathbf{i})^{-1}$, $(H_0 + \mathbf{i})^{-1}g_\delta$ are Hilbert-Schmidt operators and $f_\delta(H_0 + \mathbf{i})^{-2}$ is a trace one. Since $g_\delta^{-1}Vg_\delta^{-1}, Vf_\delta^{-1} \in L^\infty$, it follows that

$$(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} = (H_0 + \mathbf{i})^{-1}g_\delta[g_\delta^{-1}Vg_\delta^{-1}]g_\delta(H_0 + \mathbf{i})^{-1}$$

and $V(H_0 + \mathbf{i})^{-2}$ are trace class operators. Next we write

$$(H + \mathbf{i})^{-1} - (H_0 + \mathbf{i})^{-1} = -(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} + (H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}$$

and conclude that $(H + \mathbf{i})^{-1} - (H_0 + \mathbf{i})^{-1} = -(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}$ is trace class. Now consider the following equalities

$$\begin{aligned} (\mathbf{i} + H)^{-1}V(\mathbf{i} + H)^{-1} &= (\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1} + (\mathbf{i} + H)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1} + \\ &(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H)^{-1} + (\mathbf{i} + H)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H_0)^{-1}V(\mathbf{i} + H)^{-1} \end{aligned}$$

and

$$V(H + \mathbf{i})^{-2} = V(H_0 + \mathbf{i})^{-2} - V(H_0 + \mathbf{i})^{-1}(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1} - V(H + \mathbf{i})^{-1}V(H_0 + \mathbf{i})^{-1}(H + \mathbf{i})^{-1}.$$

By using the trace class properties established above, we get (2.13) for $z = z' = -i$. By applying the first resolvent equation

$$(H - z)^{-1} = (H + \mathbf{i})^{-1} + (\mathbf{i} - z)(H + \mathbf{i})^{-1}(H - z)^{-1},$$

we obtain the general case.

To examine $Vg(H)$, consider the function $h(x) = (x + \mathbf{i})^2g(x)$. Then $Vg(H) = V(H + \mathbf{i})^{-2}h(H)$ and since $V(H + \mathbf{i})^{-2}$ is trace class, we obtain the result. \square

For $R > 0$ introduce

$$H_R := H_0 + \chi_R(x, y)V(x, y),$$

where $\chi_R(x, y) = \chi(\frac{x}{R}, \frac{y}{R})$ with $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ in a neighborhood of $|(x, y)| \leq 1$.

Remark 1. The result of Proposition 2 concerning the trace class property of $(H - z)^{-1} - (H_0 - z)^{-1}$, $\text{Im } z \neq 0$, improves considerably Proposition 2 in [5], where much more regular potentials have been examined. On the other hand, if the potential V satisfies (1.1) and $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$, then the statements of Proposition 2 hold for the operators $(z - H_R)^{-1}V(z' - H)^{-1}$, $z \notin \mathbb{R}, z' \notin \mathbb{R}$.

The proof of Theorem 1 in the general case will be a simple consequence of the following

Lemma 5. Let $V(x, y)$ be as in Theorem 1. Then for $f \in C_0^\infty(\mathbb{R})$ we have

$$\lim_{R \rightarrow \infty} \text{tr} \left(f(H_R) - f(H) \right) = 0, \quad (2.14)$$

$$\lim_{R \rightarrow \infty} \text{tr} \left(\partial_x(\chi_R V) f(H_R) \right) = \text{tr} \left(\partial_x V f(H) \right). \quad (2.15)$$

Proof. Let $g(x) = (x + \mathbf{i})f(x)$ be as above. We decompose

$$f(H_R) - f(H) = \left((H_R + \mathbf{i})^{-1} - (H + \mathbf{i})^{-1} \right) g(H) + (H_R + \mathbf{i})^{-1} \left(g(H_R) - g(H) \right) = J_R + K_R.$$

From the first resolvent identity, we obtain

$$J_R = (H_R - \mathbf{i})^{-1}(1 - \chi_R)V(H + \mathbf{i})^{-1}g(H) = (H_R - \mathbf{i})^{-1}(1 - \chi_R)Vf(H).$$

According to Proposition 2, the operator $Vf(H)$ is trace class and $(H_R - \mathbf{i})^{-1}(1 - \chi_R)$ converges strongly to zero. Then from Proposition 1 it follows that

$$\lim_{R \rightarrow \infty} \text{tr } J_R = 0. \quad (2.16)$$

To treat $\text{tr } K_R$, as in the proof of Lemma 4, we use the Helffer-Sjöstrand formula and write

$$\begin{aligned} \text{tr } K_R &= -\frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) \text{tr} \left((H_R + \mathbf{i})^{-1} \left((z - H_R)^{-1} - (z - H)^{-1} \right) \right) L(dz) \\ &= \frac{1}{\pi} \int \bar{\partial} \tilde{g}(z) \text{tr} \left((H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V(z - H)^{-1} \right) L(dz). \end{aligned}$$

By cyclicity of the traces we obtain

$$\begin{aligned} \text{tr} \left((H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V(z - H)^{-1} \right) &= \text{tr} \left((z - H_R)^{-1} (1 - \chi_R) V(z - H)^{-1} (H_R + \mathbf{i})^{-1} \right) \\ &= \text{tr} \left((z - H_R)^{-1} (1 - \chi_R) V(z - H)^{-1} (H + \mathbf{i})^{-1} \right) \\ &\quad + \text{tr} \left((1 - \chi_R) V(H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R) V(z - H)^{-1} (H + \mathbf{i})^{-1} \right). \end{aligned}$$

Now notice that for $z \notin \mathbb{R}$ the operators $(1 - \chi_R) V(H_R + \mathbf{i})^{-1} (z - H_R)^{-1} (1 - \chi_R)$ and $(z - H_R)^{-1} (1 - \chi_R)$ converge strongly to zero. On the other hand, from Proposition 2 we deduce that the operator $V(z - H)^{-1} (\mathbf{i} + H)^{-1}$ is trace class. Thus for $z \notin \mathbb{R}$, we conclude that the integrand converge to 0 as $R \rightarrow \infty$. An application of the Lebesgue convergence domination theorem combined with the estimates (2.13) yield

$$\lim_{R \rightarrow \infty} \text{tr } K_R = 0. \quad (2.17)$$

Putting together (2.16) and (2.17), we obtain (2.14).

Next, we pass to the proof of (2.15). A simple calculus shows that

$$\partial_x (\chi_R V) f(H_R) = \partial_x (\chi_R V) (f(H_R) - f(H)) + \frac{1}{R} (\partial_x \chi)_R V f(H) + (\chi_R \partial_x V f(H)). \quad (2.18)$$

Repeating the same arguments as in the proof of (2.14), we show that

$$\lim_{R \rightarrow \infty} \text{tr} \left(\partial_x (\chi_R V) (f(H_R) - f(H)) \right) = 0. \quad (2.19)$$

On the other hand, since $\frac{1}{R} (\partial_x \chi)_R$ (resp. χ_R) converges strongly to zero (resp. 1), it follows from Proposition 1 that

$$\lim_{R \rightarrow \infty} \text{tr} \left(\frac{1}{R} (\partial_x \chi)_R V f(H) \right) = 0, \quad \lim_{R \rightarrow \infty} \text{tr} \left(\chi_R \partial_x V f(H) \right) = \text{tr} \left(\partial_x V f(H) \right),$$

which together with (2.18) and (2.19) yield (2.15). \square

End of the proof of Theorem 1. Applying Theorem 1 to H_R , we obtain :

$$\text{tr} \left[f(H_R) - f(H) \right] + \text{tr} \left[f(H) - f(H_0) \right] = \text{tr} \left[f(H_R) - f(H_0) \right] = -\text{tr} \left(\partial_x (\chi_R V) f(H) \right),$$

and an application of Lemma 5 implies Theorem 1.

3. MOURRE ESTIMATE AND EMBEDDED EIGENVALUES

Consider the operator

$$Q = (D_x - By)^2 + D_y^2 + V(x, y),$$

and set $\langle x \rangle = (1 + |x|^2)^{1/2}$, $\langle D_x \rangle = (1 + D_x^2)^{1/2}$.

Lemma 6. *Assume that $V, \partial_x V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ and let $\|\mathbb{I}_{\{|x|+|y|>R\}}(x, y) \partial_x V\|_{L^\infty} \rightarrow 0$ for $R \rightarrow +\infty$. Then for all $f \in C_0^\infty(\mathbb{R})$, the operator $f(H) \partial_x V f(H)$ is compact.*

Proof. Let $\varphi(x, y) \in C_0^\infty(\mathbb{R}^2)$ be equal to one near zero. Set $\varphi_n(x, y) = \varphi(\frac{x}{n}, \frac{y}{n})$. According to Lemma 3, the operator $f(H) \varphi_n \partial_x V f(H)$ is trace class. The set of compact operators is closed with respect to the norm $\|\cdot\|_{\mathcal{L}(L^2)}$ and the lemma follows from the obvious estimate

$$\|f(H)(1 - \varphi_n) \partial_x V f(H)\|_{\mathcal{L}(L^2)} \leq \|f^2(H)\|_{\mathcal{L}(L^2)} \|(1 - \varphi_n) \partial_x V\|_\infty.$$

□

Theorem 3. *Let $[a, b] \subset \mathbb{R}$. Under the assumptions of Lemma 6, there exists a compact operator K such that*

$$\mathbb{I}_{[a,b]}(H) [\partial_x, H] \mathbb{I}_{[a,b]}(H) \geq \epsilon \mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]}(H) K \mathbb{I}_{[a,b]}(H). \quad (3.1)$$

Proof. Since the operator ∂_x commutes with $(D_x - By)$ and D_y^2 , we have $[\partial_x, H] = \epsilon + \partial_x V$. Consequently,

$$\begin{aligned} \mathbb{I}_{[a,b]}(H) [\partial_x, H] \mathbb{I}_{[a,b]}(H) &= \epsilon \mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]}(H) \partial_x V \mathbb{I}_{[a,b]}(H) \\ &= \epsilon \mathbb{I}_{[a,b]}(H) + \mathbb{I}_{[a,b]} f(H) \partial_x V f(H) \mathbb{I}_{[a,b]}(H), \end{aligned} \quad (3.2)$$

where $f \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $f = 1$ on $[a, b]$. Thus, Theorem 3 follows from Lemma 6. □

The use of commutators with the operator ∂_x is well known for the analysis of the operator without magnetic field ($B = 0$) (see the pioneering work [2] and [1] for a more complete list of references). On the other hand, to treat crossed magnetic and electric fields we need Lemma 1 and Lemma 3.

Corollary 1. *In addition to the assumptions of Theorem 3 assume that $\partial_x^2 V \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then the point spectrum of H in $[a, b]$ is finite and with finite multiplicity. Moreover, the singular continuous spectrum of H is empty.*

Proof. Set $A = D_x$ and let $\alpha \in \mathbb{R}$. The explicit formula

$$e^{i\alpha A} (H + \mathbf{i})^{-1} = (e^{i\alpha A} H e^{-i\alpha A} + \mathbf{i})^{-1} e^{i\alpha A} = (H + \epsilon\alpha + V(x + \alpha, y) - V(x, y) + \mathbf{i})^{-1} e^{i\alpha A}$$

shows that $e^{i\alpha A}$ leaves $D(H)$ invariant. On the other hand, since

$$\begin{aligned} \|H e^{i\alpha A} (H + \mathbf{i})^{-1} \psi\| &= \|e^{-i\alpha A} H e^{i\alpha A} (H + \mathbf{i})^{-1} \psi\| \\ &= \left\| \left(H - \epsilon\alpha + V(x - \alpha, y) - V(x, y) \right) (H + \mathbf{i})^{-1} \psi \right\|, \end{aligned}$$

we deduce that for each $\varphi \in D(H)$

$$\sup_{|\alpha| < 1} \|H e^{i\alpha A} \varphi\| < \infty.$$

Combining this with the fact $\mathbf{i}[A, H] = \epsilon + \partial_x V$, $[A, [A, H]] = -\partial_x^2 V$ and using (3.1), we conclude that the self-adjoint operator A is a conjugate operator for H at every $E \in \mathbb{R}$ in the sense of [14]. Consequently, Corollary 1 follows from the main result in [14] (see also [1], [6]).

□

Remark 2. For any sign-definite and bounded potential $V(x, y)$ such that $|V(x, y)| \rightarrow 0$ as $|x| + |y| \rightarrow \infty$ sufficiently fast in [15] and [13] it was established that for $\epsilon = 0$ the potential V creates an infinite number of eigenvalues of Q which accumulate to Landau levels. The above corollary shows that only a finite number of these eigenvalues may survive in the presence of a non vanishing constant electric field. In general, the problem of absence of embedded eigenvalues when $\epsilon \neq 0$ remains open and this is an interesting conjecture.

For a fixed value of $\epsilon \neq 0$, the following result shows that there are potentials for which H has absolutely continuous spectrum without embedded eigenvalues.

Corollary 2. Fix $\epsilon > 0$. Assume that $\partial_x^\alpha V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$, $\alpha = 0, 1, 2$ and

$$\epsilon + \partial_x V(x, y) > c > 0, \quad (3.3)$$

uniformly on $(x, y) \in \mathbb{R}^2$. Then H has no eigenvalues. Moreover, for $s > 1/2$, the following estimates holds uniformly on λ in a compact interval

$$\|\langle D_x \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle D_x \rangle^{-s}\| = \mathcal{O}_\epsilon(1). \quad (3.4)$$

Proof. Let $[a, b]$ be a compact interval in \mathbb{R} . From (3.1) and (3.3), we have

$$\mathbb{I}_{[a, b]}(H) [\partial_x, H] \mathbb{I}_{[a, b]}(H) \geq c \mathbb{I}_{[a, b]}(H). \quad (3.5)$$

According to the proof of Corollary 1, $A = D_x$ is a conjugate operator in the sense of [14]. Combining this with (3.5) we deduce from [14] that H has no eigenvalue in \mathbb{R} . Applying once more Mourre theorem (see [14], [1], [6]), we obtain the estimate (3.4). □

4. LIMITING ABSORPTION PRINCIPLE

In this section we treat the case when ϵ is small enough. Notice that when ϵ tends to zero in general the assumption $\epsilon + \partial_x V > c > 0$ is not satisfied and we cannot apply Corollary 2. Our goal is to study the behavior of the resolvent $(H - \lambda \pm i\delta)^{-1}$ as $\delta \rightarrow 0$ for $\lambda \notin \sigma(Q)$. For such λ we could have eigenvalues of H and a direct application of Mourre argument is not possible. We will obtain the result assuming that ϵ is small and for this purpose we need the following

Lemma 7. Assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and let $\lambda \notin \sigma(Q)$. Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$ be equal to 1 near λ and let $\text{supp } \chi \cap \sigma(Q) = \emptyset$. Then

$$\|\chi(H) \langle x \rangle^{-2}\| \leq C\epsilon^2. \quad (4.1)$$

Proof. Since $\text{supp } \chi \cap \sigma(Q) = \emptyset$, the operators $(z - Q)^{-1}$ and $(z - Q)^{-1}x(z - Q)^{-1}$ are analytic operator valued functions for z in a complex neighborhood of $\text{supp } \chi$. Let $\tilde{\chi}(z) \in C_0^\infty(\mathbb{C})$ be an almost analytic continuation of $\chi(x)$ such that

$$\bar{\partial} \tilde{\chi}(z) = \mathcal{O}(|\text{Im } z|^\infty)$$

and $\text{supp } \tilde{\chi}(z) \cap \sigma(Q) = \emptyset$. We have the representation

$$\chi(H) = -\frac{1}{\pi} \int \bar{\partial} \tilde{\chi}(z) (z - H)^{-1} L(dz),$$

where $L(dz)$ is the Lebesgue measure in \mathbb{C} . By using the resolvent identity, we get

$$(z - H)^{-1} = (z - Q)^{-1} + \epsilon(z - Q)^{-1}x(z - Q)^{-1} + \epsilon^2(z - H)^{-1}x(z - Q)^{-1}x(z - Q)^{-1},$$

and we obtain

$$\begin{aligned}\chi(H) &= \chi(Q) - \frac{\epsilon}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z - Q)^{-1} x(z - Q)^{-1} L(dz) \\ &\quad - \frac{\epsilon^2}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z - H)^{-1} x(z - Q)^{-1} x(z - Q)^{-1} L(dz).\end{aligned}$$

Since $\text{supp } \tilde{\chi}(z) \cap \sigma(Q) = \emptyset$, the first two terms on the right hand side vanish. Consequently,

$$\chi(H) = -\frac{\epsilon^2}{\pi} \int \bar{\partial} \tilde{\chi}(z)(z - H)^{-1} x(z - Q)^{-1} x(z - Q)^{-1} L(dz). \quad (4.2)$$

Next we observe that

$$x(z - Q)^{-1} = (z - Q)^{-1} x + (z - Q)^{-1} [x, Q](z - Q)^{-1} = (z - Q)^{-1} x + L_1.$$

We have $[x, Q] = 2(D_x - By)$. Thus it is easy to see that for $z \notin \sigma(Q)$, $L_1 = (z - Q)^{-1} [x, Q](z - Q)^{-1}$ is a bounded operator since $(D_x - By)(\mathbf{i} - Q)^{-1}$ is bounded and $(z - Q)^{-1} = (\mathbf{i} - Q)^{-1} + (\mathbf{i} - Q)^{-1}(\mathbf{i} - z)(z - Q)^{-1}$. We write

$$\begin{aligned}x(z - Q)^{-1} x(z - Q)^{-1} &= (z - Q)^{-1} x(z - Q)^{-1} x \\ &\quad + (z - Q)^{-1} x L_1 + L_1 (z - Q)^{-1} x + L_1^2 = \sum_{j=1}^4 I_j.\end{aligned}$$

The operators $I_4 = L_1^2$ and $I_3 = L_1(z - Q)^{-1} x \langle x \rangle^{-2}$ are bounded. To see that $I_1 \langle x \rangle^{-2}$ is bounded, note that

$$I_1 \langle x \rangle^{-2} = (z - Q)^{-2} x^2 \langle x \rangle^{-2} + (z - Q)^{-1} L_1 x \langle x \rangle^{-2}.$$

Finally,

$$I_2 \langle x \rangle^{-2} = (z - Q)^{-2} x [x, Q](z - Q)^{-1} \langle x \rangle^{-2} + (z - Q)^{-1} L_1 [x, Q](z - Q)^{-1} \langle x \rangle^{-2}$$

and since the second term on the right hand side is bounded, it remains to examine the operator

$$x [x, Q](z - Q)^{-1} \langle x \rangle^{-2} = [x, Q] x (z - Q)^{-1} \langle x \rangle^{-2} + 2\mathbf{i} (z - Q)^{-1} \langle x \rangle^{-2}.$$

Applying the above argument, we see that the last operator is bounded. Consequently, the operator under integration in (4.2) is bounded by $\mathcal{O}(|\text{Im } z|^{-1})$ and this proves the statement. \square

Proposition 3. Assume that $\partial_x^\alpha V \in C^0(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2; \mathbb{R})$ for $\alpha = 0, 1, 2$ and let $\langle x \rangle^2 \partial_x V \in L^\infty(\mathbb{R}^2)$. Let $[a, b]$ be a compact interval such that $[a, b] \cap \sigma(Q) = \emptyset$. Then for $s > 1/2$ and sufficiently small $\epsilon_0 > 0$ we have the following estimate uniformly with respect to $\lambda \in [a, b]$ and $\epsilon \in]0, \epsilon_0]$

$$\|\langle D_x \rangle^{-s} (H - \lambda \pm \mathbf{i}0)^{-1} \langle D_x \rangle^{-s}\| \leq C\epsilon^{-1}. \quad (4.3)$$

Moreover, H has no embedded eigenvalues and singular continuous spectrum in $[a, b]$.

Proof. Let $[a - \delta, b + \delta] \cap \sigma(Q) = \emptyset$ for $0 < \delta \ll 1$. Choose a function $\chi(t) \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that $\text{supp } \chi \subset [a - \delta, b + \delta]$ and $\chi(t) = 1$ for $a_1 = a - \delta/2 \leq t \leq b + \delta/2 = b_1$. Then

$$\begin{aligned}\mathbb{I}_{[a_1, b_1]}(H) [\partial_x, H] \mathbb{I}_{[a_1, b_1]}(H) &= \epsilon \mathbb{I}_{[a_1, b_1]}(H) + \mathbb{I}_{[a_1, b_1]}(H) \partial_x V \mathbb{I}_{[a_1, b_1]}(H) \\ &= \epsilon \mathbb{I}_{[a_1, b_1]}(H) + \mathbb{I}_{[a_1, b_1]}(H) \left(\chi(H) \langle x \rangle^{-2} \right) \left(\langle x \rangle^2 \partial_x V \right) \mathbb{I}_{[a_1, b_1]}(H)\end{aligned}$$

Our assumption implies that the multiplication operator $\langle x \rangle^2 \partial_x V \in L^\infty$, while Lemma 7 says that

$$\|\chi(H) \langle x \rangle^{-2}\| \leq C\epsilon^2.$$

Thus

$$\mathbb{I}_{[a_1, b_1]}(H) \left(\chi(H) \langle x \rangle^{-2} \right) \left(\langle x \rangle^2 \partial_x V \right) \mathbb{I}_{[a_1, b_1]}(H) \leq C_1 \epsilon^2 \mathbb{I}_{[a_1, b_1]}(H)$$

and with a constant $c_0 > 0$ we deduce

$$\mathbb{I}_{[a_1, b_1]}(H) [\partial_x, H] \mathbb{I}_{[a_1, b_1]}(H) \geq c_0 \epsilon \mathbb{I}_{[a_1, b_1]}(H).$$

Then it is well known (see for instance [14], [1], [6]) that for $\lambda \in [a, b]$ we get (4.3) and H has no eigenvalues and singular continuous spectrum in $[a, b]$. \square

Remark 3. As we mentioned in Remark 2 for sign-definite rapidly decreasing potentials the spectrum of the operator Q is formed by infinite number eigenvalues having as points of accumulation the Landau levels $\mu_n = (2n+1)B$, $n \in \mathbb{N}$. For such potentials Proposition 3 shows that the embedded eigenvalues of H could appear only in small neighborhoods of the eigenvalues of Q . Since in every interval we may have only a finite number of eigenvalues of H , it is clear that for some eigenvalues ν of Q there are no eigenvalues of H in their neighborhoods. Moreover, it was proved in [12] that for potentials $V \in C_0^\infty(\mathbb{R}^2)$ we have $\sigma(Q) \cap [\mu_n - B, \mu_n + B] \subset (\mu_n - Cn^{-1/2}, \mu_n + Cn^{-1/2})$, $n \geq N$ with $C > 0$ and N depending only on $\sup |V|$ and the diameter of the support of V . Thus for M large the embedded eigenvalues $\lambda \geq M$ of H are sufficiently close to Landau levels Λ_n .

5. ESTIMATES FOR THE DERIVATIVE OF THE SPECTRAL SHIFT FUNCTION

First we notice that the assumption (1.4) makes possible to define the spectral shift function $\xi(\lambda, \epsilon)$ related to operators $H_0(\epsilon) = H_0(B, \epsilon)$ and $H(\epsilon) = H_0(B, \epsilon) + V(x, y)$ by the equality

$$\langle \xi', f \rangle = \text{tr} \left(f(H(\epsilon)) - f(H_0(\epsilon)) \right), \quad f \in C_0^\infty(\mathbb{R}).$$

Here and below we omit the dependence of B in the notations. Our purpose in this section is to establish Theorem 2. For the proof we need the following

Proposition 4. Under the assumptions of Theorem 2, for $\lambda_0 \notin \sigma(Q)$ and $1/2 < s < \min(1/2 + \delta/4, 1)$ the operator

$$\langle D_x \rangle^s \partial_x V \left[(Q - z)^{-1} x \right]^n \langle D_x \rangle^s$$

is trace class for z in a small complex neighborhood $\Xi \subset \mathbb{C}$ of λ_0 .

Proof. Before starting the proof, notice that it is easy to establish the statement for $z \ll 0$ since in this case the operator $(Q - z)^{-1}$ is a pseudodifferential one and we can apply the calculus of pseudodifferential operators and the criteria which guarantees that a pseudodifferential operator is trace class (see for instance, [3], Theorem 9.4). For $z \in \mathbb{R}^+ \setminus \sigma(Q)$ this is not the case and $(Q - z)^{-1}$ is a bounded operator but not a pseudodifferential one. We may replace $(Q - z)^{-1}$ by the pseudodifferential operator $(Q - \mathbf{i})^{-1}$ modulo bounded operators but therefore it is difficult to examine the product involving many bounded operators and factors x^k . To overcome this difficulty, we are going to apply a convenient decomposition by product of operators having in mind that the operator on the left of a such product must be trace class one.

First we treat the case $n = 2$, the general case will be covered by a recurrence. We start with the analysis of the operator

$$\langle D_x \rangle^{2s} \partial_x V [(Q - z)^{-1} x]^2. \tag{5.1}$$

Our goal is to show that (5.1) is a trace class operator. Write

$$\langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 \langle x \rangle^{-2} (Q - z)^{-1} x (Q - z)^{-1} x = \langle D_x \rangle^{2s} (\partial_x V) \langle x \rangle^2 (Q - z)^{-1} \langle x \rangle^{-2} x (Q - z)^{-1} x$$

$$\begin{aligned}
& + \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-1} [Q, \langle x \rangle^{-2}] (Q - z)^{-1} x (Q - z)^{-1} x \\
& = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} \left[\langle x \rangle^{-2} x^2 + [Q, \langle x \rangle^{-2} x] (Q - z)^{-1} x \right] \\
& + \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-1} [Q, \langle x \rangle^{-2}] (Q - z)^{-1} x (Q - z)^{-1} x = T_1 + T_2.
\end{aligned}$$

To deal with T_1 , we use the representation

$$T_1 = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} W_1$$

and we will show that the operator

$$\begin{aligned}
W_1 & = \langle x \rangle^{-2} x^2 + [Q, \langle x \rangle^{-2} x] (Q - z)^{-1} x \\
& = \langle x \rangle^{-2} x^2 - \mathbf{i} \left[(D_x - By) \frac{1 - x^2}{(1 + x^2)^2} + \frac{1 - x^2}{(1 + x^2)^2} (D_x - By) \right] (Q - z)^{-1} x
\end{aligned}$$

is bounded. Consider the operator

$$\begin{aligned}
(D_x - By) \frac{(1 - x^2)}{(1 + x^2)^2} (Q - z)^{-1} x & = (D_x - By) \frac{(1 - x^2)x}{(1 + x^2)^2} (Q - \mathbf{i})^{-1} \left[1 + (z - \mathbf{i})(Q - z)^{-1} \right] \\
& + (D_x - By) \frac{1 - x^2}{(1 + x^2)^2} (Q - z)^{-1} [Q, x] (Q - z)^{-1}.
\end{aligned}$$

The pseudodifferential operator

$$(D_x - By) \frac{(1 - x^2)x}{(1 + x^2)^2} (Q - \mathbf{i})^{-1}$$

is bounded and the product of this operator with $\left[1 + (\mathbf{i} - z)(Q - z)^{-1} \right]$ is bounded, too. As in the proof of Lemma 7, we see that $[Q, x](Q - z)^{-1}$ is bounded and with the same argument we treat the other terms. Thus we conclude that W_1 is a bounded operator. Next we write

$$T_2 = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2} W_2,$$

where

$$W_2 = [Q, \langle x \rangle^{-2}] x (Q - z)^{-1} x + \left[Q, [Q, \langle x \rangle^{-2}] \right] (Q - z)^{-1} x (Q - z)^{-1} x = W_{21} + W_{22}.$$

We have

$$W_{21} = 2\mathbf{i} \left[(D_x - By) \frac{x^2}{(1 + x^2)^2} (Q - z)^{-1} x + \frac{x}{(1 + x^2)^2} (D_x - By) x (Q - z)^{-1} x \right]$$

and as above we deduce that W_{21} is a bounded operator. For the analysis of W_{22} , we write

$$\begin{aligned}
W_{22} & = \left\{ \frac{1 - 3x^2}{(1 + x^2)^3} 4(D_x - By)^2 + R_1(x)(D_x - By) + R_2(x) + \frac{x}{(1 + x^2)^2} (4\partial_x V + 8BD_y) \right\} \\
& \quad \times (Q - z)^{-1} x (Q - z)^{-1} x.
\end{aligned}$$

A simple calculus gives

$$\begin{aligned}
& (Q - z)^{-1} x (Q - z)^{-1} x = (Q - z)^{-1} x^2 (Q - z)^{-1} + (Q - z)^{-1} x M_1 \\
& = x^2 (Q - z)^{-2} + 4(Q - z)^{-1} x (D_x - By) (Q - z)^{-2} + x (Q - z)^{-1} M_1 + (Q - z)^{-1} M_2 \\
& = x^2 (Q - z)^{-2} + 4x (Q - z)^{-1} M_3 + (Q - z)^{-1} M_4 \\
& = x^2 (Q - \mathbf{i})^{-2} M_5 + 4x (Q - \mathbf{i})^{-1} M_6 + (Q - \mathbf{i})^{-1} M_7,
\end{aligned}$$

where $M_k, k = 1, 2, \dots$, denote bounded operators. The pseudodifferential calculus implies that the product of the term in the brackets $\{\dots\}$ with $x^j(Q - \mathbf{i})^{-j}$, $j = 1, 2$ is a bounded operator. Combining this with the above equality, we conclude that W_{22} is bounded.

Now it remains to see that the operator

$$\mathcal{T} = \langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - z)^{-2}$$

is trace class. For this purpose we replace $(Q - z)^2$ by

$$(Q - \mathbf{i})^{-2} \left[I + (z - \mathbf{i})(Q - z)^{-1} \right]^2$$

and consider the pseudodifferential operator

$$\langle D_x \rangle^{2s} \partial_x V \langle x \rangle^2 (Q - \mathbf{i})^{-2} \quad (5.2)$$

with principal symbol

$$g_s(x, y, \xi, \eta) = \frac{\xi^{2s} (\partial_x V)(x, y) (1 + x^2)}{\left((\xi - By)^2 + \eta^2 + V(x, y) - \mathbf{i} \right)^2}.$$

We use the estimate $\langle \xi \rangle^{2s} \leq C \langle \xi - By \rangle^{2s} \langle y \rangle^{2s}$ and we apply Theorem 9.4 in [3] to deduce that (5.2) is a trace class operator. In fact we have

$$\sum_{|\alpha| \leq 5} \|\partial_{x,y,\xi,\eta}^\alpha g_s\|_{L^1(\mathbb{R}^4)} < \infty$$

since $2s < 2$ guarantees that the integral with respect to ξ is convergent, while $2s < 1 + \delta/2$ and the estimate (1.4) imply that integral with respect to y is convergent. Consequently, \mathcal{T} is a trace class operator and this completes the analysis of (5.1). Notice also that the same argument implies that the operator

$$\langle D_x \rangle^s \partial_x V \left[(Q - z)^{-1} x \right]^2$$

is trace class.

To prove that the operator $\langle D_x \rangle^s \partial_x V \left[(Q - z)^{-1} x \right]^2 \langle D_x \rangle^s$ is trace class, we commute the operator $\langle D_x \rangle^s$ with $(Q - z)^{-1} x$ and $\partial_x V$ in order to reduce the proof to that of (5.1). The commutators $[x, \langle D_x \rangle^s]$ and $[V, \langle D_x \rangle^s] x$ are bounded since $s < 1$. Next

$$\begin{aligned} [(Q - z)^{-1}, \langle D_x \rangle^s] x &= (Q - z)^{-1} [V, \langle D_x \rangle^s] (Q - z)^{-1} x \\ &= (Q - z)^{-1} [V, \langle D_x \rangle^s] \left(x(Q - z)^{-1} + (Q - z)^{-1} M_1 \right) = (Q - z)^{-1} M_2 \end{aligned}$$

and we obtain operators which can be handled by the above argument. Thus the assertion is proved for $n = 2$.

Passing to the general case $n > 2$, assume that the assertion holds for $n = 2, \dots, k - 1$, and suppose that V satisfy the estimate (1.4) with $n = k$. The idea is to replace the operator

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^k \langle D_x \rangle^s$$

by the trace class operator $\langle D_x \rangle^s (\partial_x V) x^k (Q - z)^{-2} \langle D_x \rangle^s$ plus a sum of several operators which are trace class according to the recurrence assumption. Notice that if M_j is bounded operator obtained

as a product of $(D_x - By)$ and $(Q - z)^{-j}$, $j \geq 1$, the operator $\langle D_x \rangle^{-s} M_j \langle D_x \rangle^s$ becomes a bounded operators and this makes possible to exploit the representation

$$\langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots M_j \langle D_x \rangle^s = \left[\langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots \langle D_x \rangle^s \right] \left(\langle D_x \rangle^{-s} M_j \langle D_x \rangle^s \right)$$

Thus we reduce the analysis to the trace class property of $\langle D_x \rangle^s \partial_x V (Q - z)^{-1} x \dots \langle D_x \rangle^s$. For simplicity of the notations we will write $A \sim_t B$ if the difference $A - B$ is a trace class operator.

We start with the observation that

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^k \langle D_x \rangle^s \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s.$$

We can establish this by a recurrence. For $k - 1$ we apply the equality

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-1} \langle D_x \rangle^s &= \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s \\ &\quad \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} [Q, x] (Q - z)^{-1} \langle D_x \rangle^s \\ &\sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s. \end{aligned}$$

Commuting $(Q - z)^{-1}$ and x^2 , we obtain the result for $k - 1$ and in the same way we continue for $p \leq k - 1$.

Next we commute $(Q - z)^{-1}$ and x^2 and get

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-2} (Q - z)^{-1} x^2 (Q - z)^{-1} \langle D_x \rangle^s \\ \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-3} (Q - z)^{-1} x^3 (Q - z)^{-2} \langle D_x \rangle^s. \end{aligned}$$

Indeed, $[Q, x^2] = 4(D_x - By)x = -4\mathbf{i}x(D_x - By) - 2$ yields

$$(Q - z)^{-1} x^2 (Q - z)^{-1} = x^2 (Q - z)^{-2} - 4\mathbf{i} (Q - z)^{-1} x (D_x - By) (Q - z)^{-1} - 2(Q - z)^{-2}$$

and for the term

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-1} (D_x - By) (Q - z)^{-1} \langle D_x \rangle^s$$

we use the recurrence assumption and the fact that $M_2 = (D_x - By)(Q - z)^{-1}$ is a bounded operator. In the same way for $1 \leq j \leq k - 1$ we show that

$$\begin{aligned} \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-j} (Q - z)^{-1} x^j (Q - z)^{-2} \langle D_x \rangle^s \\ \sim_t \langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^{k-j-1} (Q - z)^{-1} x^{j+1} (Q - z)^{-2} \langle D_x \rangle^s, \end{aligned}$$

taking into account the equality

$$[Q, x^j] = 2j(D_x - By)x^{j-1} = 2jx^{j-1}(D_x - By) - 2\mathbf{i}j(j-1)x^{j-1}$$

and the recurrence assumption. Finally, we prove that

$$\langle D_x \rangle^s \partial_x V [(Q - z)^{-1} x]^k \langle D_x \rangle^s \sim_t \langle D_x \rangle^s (\partial_x V) x^k (Q - z)^{-2} \langle D_x \rangle^s$$

and, as in the proof in the case $n = 2$, we conclude that the operator on the right hand side is trace class one. □

After this preparation we pass to the proof of Theorem 2.

Proof of Theorem 2. Let $\Xi \subset \mathbb{R}$ be a small neighborhood of λ_0 such that $\Xi \cap \sigma(Q) = \emptyset$. For the simplicity of the notations we will write $H(\epsilon)$, $\xi(\lambda, \epsilon)$ instead of $H(B, \epsilon)$, $\xi(\lambda; B, \epsilon)$. Given $f \in C_0^\infty(\Xi)$, introduce an almost analytic continuation $\tilde{f} \in C_0^\infty(\mathbb{C})$ of f so that $\bar{\partial}\tilde{f}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$ and $\operatorname{supp} \tilde{f}(z) \cap \sigma(Q) = \emptyset$. Since $(z - Q)^{-1}$ is analytic over the support of $\tilde{f}(z)$, applying the resolvent equality, we get

$$\begin{aligned} \partial_x V f(H(\epsilon)) &= -\frac{1}{\pi} \int \bar{\partial}\tilde{f}(z) \partial_x V (z - H(\epsilon))^{-1} L(dz) \\ &= (-1)^{n+1} \frac{\epsilon^n}{\pi} \int \bar{\partial}\tilde{f}(z) \partial_x V [(z - Q)^{-1} x]^n (z - H(\epsilon))^{-1} L(dz). \end{aligned} \quad (5.3)$$

Taking into account Proposition 4 and the cyclicity of the trace, we get

$$\begin{aligned} &\operatorname{tr} \int \bar{\partial}\tilde{f}(z) \langle D_x \rangle^{-s} \left[\langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s \right] \langle D_x \rangle^{-s} (z - H(\epsilon))^{-1} L(dz) \\ &= \operatorname{tr} \int \bar{\partial}\tilde{f}(z) \left[\langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s \right] \langle D_x \rangle^{-s} (z - H(\epsilon))^{-1} \langle D_x \rangle^{-s} L(dz). \end{aligned}$$

Set $W(z) = \langle D_x \rangle^s \partial_x V [(z - Q)^{-1} x]^n \langle D_x \rangle^s$ and note that for $z \in \operatorname{supp} \tilde{f}$ this operator is trace class and $W(z)$ is analytic. We write

$$\begin{aligned} &-\frac{1}{\pi} \int \bar{\partial}\tilde{f}(z) \operatorname{tr} \left(\partial_x V [(z - Q)^{-1} x]^n (z - H(\epsilon))^{-1} \right) L(dz) \\ &= \frac{1}{\pi} \lim_{\eta \searrow 0} \left[\int_{\operatorname{Im} z > 0} \bar{\partial}\tilde{f}(z + i\eta) \operatorname{tr} \left[\left(W(z + i\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z + i\eta))^{-1} \langle D_x \rangle^{-s} \right) \right] L(dz) \right. \\ &\quad \left. + \int_{\operatorname{Im} z < 0} \bar{\partial}\tilde{f}(z - i\eta) \operatorname{tr} \left(W(z - i\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z - i\eta))^{-1} \langle D_x \rangle^{-s} \right) L(dz) \right]. \end{aligned}$$

Notice that the functions

$$\operatorname{tr} \left(W(z \pm i\eta) \langle D_x \rangle^{-s} (H(\epsilon) - (z \pm i\eta))^{-1} \langle D_x \rangle^{-s} \right)$$

are analytic in $\pm \operatorname{Im} z > 0$. Applying Green formula, as in Lemma 1 in [4], we deduce

$$\begin{aligned} \langle \xi'(\lambda, \epsilon), f \rangle &= \operatorname{tr} \left(f(H(\epsilon) - f(H_0)) \right) = -\frac{1}{\epsilon} \operatorname{tr} \left(\partial_x V f(H(\epsilon)) \right) \\ &= \lim_{\eta \searrow 0} \frac{(-1)^n \epsilon^{n-1}}{2\pi i} \int f(\lambda) \operatorname{tr} \left(W(\lambda) \left[\langle D_x \rangle^{-s} \left((H(\epsilon) - (\lambda + i\eta))^{-1} - (H(\epsilon) - (\lambda - i\eta))^{-1} \right) \langle D_x \rangle^{-s} \right] \right) d\lambda, \end{aligned}$$

where the integral is taken in the sense of distributions. On the other hand, Proposition 4 combined with (4.3) show that the right hand side of the above representation is finite and has order $\mathcal{O}(\epsilon^{n-2})$. Thus for $\forall f \in C_0^\infty(\Xi)$ we obtain

$$\langle \xi'(\lambda, \epsilon), f \rangle = \int f(\lambda) T_\epsilon(\lambda) d\lambda$$

with $T_\epsilon(\lambda) = \mathcal{O}(\epsilon^{n-2})$ and this completes the proof.

6. APPENDIX

The proof of the following Lemma is similar to the proof of Proposition 2.1 in [5] and for the reader convenience we give it.

Lemma 8. *Let $\delta > 0$ and let $k_j(x, y) = \langle x \rangle^{-j(1+\delta)} \langle y \rangle^{-j(\frac{1}{2}+\delta)}$, $j = 1, 2$. The operators $G_2 := k_2(H_0 + \mathbf{i})^{-2}$, G_2^* , (resp. $G_1 := k_1(H_0 + \mathbf{i})^{-1}$, G_1^*), are trace class (resp. Hilbert-Schmidt).*

Proof. Without loss of the generality we may assume that $B = \epsilon = 1$. Introduce the unitary operator $U : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ by

$$(Uu)(x, y) = \frac{2}{\pi} \iint_{\mathbb{R}^2} e^{i\varphi(x, y, x', y')} u(x', y') dx' dy',$$

where $\varphi(x, y, x', y') = xy - xy' - x'y + x'y' - \frac{1}{2}y'$. A simple calculus shows that

$$\tilde{H}_0 = U^{-1}H_0U = (D_y^2 + y^2) + x - \frac{1}{4},$$

$$\tilde{k}_j^\omega = U^{-1}k_jU = k_j^\omega\left(x - D_y - \frac{1}{2}, y + D_x\right).$$

Since U is unitary, it suffices to prove the lemma for $\tilde{G}_j := UG_jU^{-1} = \tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j}$.

Let $\chi(t) \in C_0^\infty(\mathbb{R}; [0, 1])$ be a cut-off function such that $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$. Fix a number k , $\max\{1, \frac{2}{1+2\delta}\} < k < 2$, and introduce the symbol

$$q(x, y, \eta) = \chi\left(\frac{\langle y, \eta \rangle^k}{|\eta^2 + y^2 + (x + i)|}\right),$$

where $\langle y, \eta \rangle = (1 + y^2 + \eta^2)^{1/2}$. It clear that $q(x, y, \eta) \in S^0(\mathbb{R}_{(x, \xi, y, \eta)}^4)$ and we set $A = q^\omega(x, y, D_y)$. We decompose

$$\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} = A\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} + (I - A)\tilde{k}_j^\omega(\tilde{H}_0 + \mathbf{i})^{-j} = L_j + M_j. \quad (6.1)$$

To treat L_j , notice that on the support of $q(x, y, \eta)$ we have

$$(\eta^2 + y^2 + x + \mathbf{i})^{-1} \in S^0(\mathbb{R}^4; \langle y, \eta \rangle^{-k}).$$

In fact, on the support of q we obtain

$$\langle y, \eta \rangle^k \leq 2|\eta^2 + y^2 + x + \mathbf{i}|,$$

and it is easy to estimate the derivatives of $(\eta^2 + y^2 + x + \mathbf{i})^{-1}$. According to the calculus of pseudodifferential operators, L_j becomes a pseudodifferential operator with symbol in

$$S^0(\mathbb{R}^4; \langle y, \eta \rangle^{-k} \langle x - \eta \rangle^{-j(1+\delta)} \langle y + \xi \rangle^{-j(\frac{1}{2}+\delta)}),$$

and the trace norm (resp. Hilbert-Schmidt norm) of L_2 (resp. L_1) can be estimated (see for instance, Proposition 9.2 and Theorem 9.4 in [3]) by

$$\begin{aligned} \|L_1\|_{\text{HS}}^2 + \|L_2\|_{\text{tr}} &\leq C_0 \iiint \langle y, \eta \rangle^{-2k} \langle x - \eta \rangle^{-2-2\delta} \langle y + \xi \rangle^{-1-2\delta} dx d\xi dy d\eta \\ &\leq C'_0 \iint \langle y, \eta \rangle^{-2k} dy d\eta \leq C''_0. \end{aligned} \quad (6.2)$$

To deal with M_j , $j = 1, 2$, we will show that $(I - A)\tilde{k}_2^\omega$ is trace class operator and $(I - A)\tilde{k}_1^\omega$ is Hilbert-Schmidt one.

Notice that on the support of the symbol of $(I - A)$ we have

$$\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|.$$

Taking into account the estimate $\partial_x^l \partial_y^m k_j(x, y) = \mathcal{O}_{l,m}(\langle x \rangle^{-j(1+\delta)} \langle y \rangle^{-j(\frac{1}{2}+\delta)})$, we get

$$\begin{aligned} \|(I - A)k_1^\omega\|_{\text{HS}}^2 + \|(I - A)k_2^\omega\|_{\text{tr}} &\leq C_1 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|} \langle x - \eta \rangle^{-2-2\delta} \langle y + \xi \rangle^{-1-2\delta} dx d\xi dy d\eta \quad (6.3) \\ &\leq C_2 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + x + \mathbf{i}|} \langle x - \eta \rangle^{-2-2\delta} dx dy d\eta \leq C_2 \iiint_{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u + \mathbf{i}|} \langle u \rangle^{-2-2\delta} du dy d\eta \\ &\leq C'_2 \iiint_{\substack{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u|, \\ |u| \leq \frac{1}{2} \langle y, \eta \rangle^k}} \langle u \rangle^{-2-2\delta} du dy d\eta + C'_2 \iiint_{\substack{\langle y, \eta \rangle^k \geq |\eta^2 + y^2 + \eta + u|, \\ |u| \geq \frac{1}{2} \langle y, \eta \rangle^k}} \langle u \rangle^{-2-2\delta} du dy d\eta \\ &\leq C'_2 \left(\iiint_{|u| \leq C_3, |y| \leq C_3, |\eta| \leq C_3} \langle u \rangle^{-2-2\delta} du dy d\eta + \iiint_{|u| \geq \frac{1}{2} \langle y, \eta \rangle^k} \langle u \rangle^{-2-2\delta} du dy d\eta \right) \\ &\leq C_4 + C_5 \int \langle u \rangle^{-2-2\delta} \left(\int_0^{(2|u|)^{\frac{1}{k}}} r dr \right) du \leq C_4 + C_6 \int \langle u \rangle^{-2-2\delta+2/k} du \leq C_7, \end{aligned}$$

since $-2 - 2\delta + 2/k < -1$.

Using (6.1), (6.2), (6.3) and the fact that M is trace class (resp. Hilbert-Schmidt) operator if and only if M^* is trace class (resp. Hilbert-Schmidt) operator, we complete the proof of the lemma. \square

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